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# RIEMANN ZETA-FUNCTION

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# RIEMANN ZETA-FUNCTION

(INTRODUCTORY LECTURES BY K. RAMACHANDRA)

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# DEDICATION

Professor GODFREY HAROLD HARDY (1877–1947) the most unforgettable mathematician of this century was born in U.K. on 7–2–1877. He has established a profound school of thought throughout the WORLD and in particular in INDIA. INDIA owes a great debt of gratitude to this mathematician for bringing to lime-light its great genius SRINIVASA RAMANUJAN. I have the honour of dedicating these lectures to commemorate the birthday centenary of Professor G. H. HARDY.

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*Dated* 9–12–1977.

# EDITOR'S NOTE

The present pamphlet is the text of national lectures delivered by Prof. K. Ramachandra at several University Centres in India, on invitation. I am thankful to Prof. Ramachandra for having made it available for publication by the Ramanujan Institute, particularly, during the Centenary of the birth of G. H. Hardy but for whom Srinivasa Ramanujan would not have come to limelight. It is but fitting for the Institute named after Ramanujan to commemorate the birth centenary, by bringing out this small volume.

I am also thankful to the authorities of the University of Madras for according the necessary permission and the University Grants Commission for having made available the necessary grants for the publication.

K. S. PADMANABHAN

# PREFACE

These lectures are meant to be a short introductory course of lectures which I (as a national lecturer 1977–78) gave in some of the Indian Universities. The subject-matter of these lectures is the distribution of the zeros of the Riemann zeta-function, the most important in the theory. The most striking development in recent times are (i) zero free regions and (ii) density results and I introduce these topics to a beginner.

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## 1. Introductory Remarks

### 1.1. *Definition* :

Let $s = \sigma + it$ be a complex variable ($\sigma, t$ real $i = +\sqrt{-1} = e^{\pi i/2}$). Put $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Observe

*Theorem* 1.1.1. The series for $\zeta(s)$ is convergent for $s = \sigma > 1$ and so it is uniformly and absolutely convergent with respect to $s$ in $\text{Re}\, s \geqslant 1 + \delta$ for every fixed $\delta > 0$. $\zeta(s)$ is therefore analytic in $\sigma > 1$.

PROOF :

$$\sum_{M+1}^{M+N} \frac{1}{n^{\sigma}} < \int_{M}^{\infty} \frac{du}{u^{\sigma}} \quad \text{for } \sigma > 1.$$

### 1.2. *Euler's product and* $\zeta(s) \neq 0$ *in* $\sigma > 1$.

As will be seen later it is important to have zero free regions for $\zeta(s)$. Practically all the known important results in this direction depend on

*Theorem* 1.2.1. Let $p$ run over all primes (i.e., irreducibles) 2, 3, 5, 7, 11,... Put

$$P(s) = \prod_{p} (1 - p^{-s})^{-1} = \prod_{p} (1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots).$$

Then $P(s)$ is (uniformly and absolutely convergent in $\sigma \geqslant 1 + \delta$ and so) analytic in $\sigma > 1$ and there

$$P(s) = \zeta(s).$$

PROOF. We have (in $\sigma > 1$) by unique factorisation theorem (which is a corollary of $p \mid ab$ implies $p \mid a$ or $p \mid b$, $m|n$ means that $m$ divides $n$, $m\, n$ is the opposite of $m|n$, we will prove this later).

$$\left| \zeta(s) - \prod_{p \leqslant N} (1 - p^{-s})^{-1} \right| < \sum_{N+1}^{\infty} \frac{1}{n^{\sigma}}$$

This proves the theorem.

*Theorem* 1.2.2. $p|ab$ implies $p|a$ or $p|b$ (provided $a$ and $b$ are positive integers).

PROOF. (by induction). The theorem is true if $p = 2$, $p = 3$ [for, if $3|a$ and $3|b$, $a = 1$ or $2$ (mod 3) and so is $b$]. Assume the truth of the theorem for $p < N$, Let $p > N$ be the least prime. Now $p|ab$ implies $p \mid |(a - px)(b - py)|$ and so $p \mid a_1 b_1$ where $0 \leqslant a_1 < p|2$ and $0 \leqslant b_1 < p|2$. If $a_1 = 0$ or $b_1 = 0$ nothing to prove. So $p|a_1 b_1$ with $1 \leqslant a_1 \leqslant (p-1)/2, 1 \leqslant b_1 \leqslant (p-1)/2$. Observe that $(p-1)/2 < p$, and so all the prime factors of $a_1$ (and similarly of $b_1$) $< p$. So we have a factorisation of the type

$$pQ = p_1 \ldots p_r \text{ with } 2 \leqslant p_i < p.$$

Now $p_1 | pQ$ and so $p_1 | p$ which is impossible or $p_1 | Q$ in which case we can cancel $p_1$ from both sides. Cancelling $p_1, \ldots, p_r$ we get

$$pQ' = 1, \text{ a contradiction.}$$

This proves the theorem.

*Theorem* 1.2.3. Let $2 \leqslant p_1 < p_2 < \ldots$ and $2 \leqslant q_1 < q_2 < \ldots$ be prime and let $p_2^{e_1} \ldots p_r^{e_r} = q_1^{f_1} \ldots q_u^{f_u}$. Then $r = u$ and $p_i = q_i (i = 1 \text{ to } r)$ and $e_i = f_i$.

PROOF. Follows on using Theorem 1.2.2 and applying

*Theorem* 1.2.4. (cancellation)

$$pa = pb$$

implies $a = b$.

PROOF. $p(a - b) = 0$ which is impossible unless $a = b$.

*Theorem* 1.2.5. $\zeta(s) \neq 0$ in $\sigma > 1$.

PROOF.

$$\left| \zeta(s) \prod_{p \leqslant N} \left(1 - \frac{1}{p^s}\right) - 1 \right| \leqslant \sum_{N+1}^{\infty} \frac{1}{n^\sigma}.$$

1.3. *Analytic continuation of* $\zeta(s)$.

*Theorem* 1.3.1. Let $\sigma > 0$. Then the series

$$f(s) = \sum_{n=1}^{\infty} \left( n^{-s} - \int_n^{n+1} \frac{du}{u^s} \right)$$

is uniformly convergent in $\sigma \geqslant \delta$, $|t| \leqslant T$ and in $\sigma > 1$ it is equal to

$$\zeta(s) - \frac{1}{s-1}.$$

PROOF.

$$n^{-s} - \int_n^{n+1} \frac{du}{u^s} = \int_0^1 du \int_0^u \frac{s\,dv}{(n+v)^{s+1}}.$$

This proves the first part and the second part is trivial.

*Theorem* 1.3.2. Let $X$ be an arbitrary positive integer $\geqslant 20(|t|+20)(K+1)$. Put

$$f_X(s) = \sum_{n=X}^{\infty}\left(n^{-s} - \int_n^{n+1}\frac{du}{u^s}\right).$$

This can be continued analytically for all $s$ and

$$|f_X(s)| < \frac{C_K}{X^{\sigma}} \text{ where } |\sigma| \leqslant K.$$

PROOF.

In $\sigma > 0$,

$$f_X(s) = s\int_0^1 du\int_0^u dv\sum_{n=0}^{\infty}\frac{1}{(n+X+v)^{s+1}}$$

and we can repeat the previous process $[K]+[(\log t)^2]$ times. Roughly every time we gain a factor $t/X$ and this proves the result. We must stop this process at the said stage since otherwise the remaining additional terms contribute too much.

1.4. *Distribution of prime numbers and the zeros of* $\zeta(s)$.

If $\zeta(s)$ has no complex zeros put $\theta = -\infty$ and if $\zeta(s)$ has, let $\theta$ denote the least upper bound of the real parts of the zeros of $\zeta(s)$. We have by Theorem 1.2.5

*Theorem* 1.4.1.

$$\theta \leqslant 1.$$

*Remark.* Looking at the present state of affairs in this direction of knowledge, even $\theta < 1$ may take several centuries. But Riemann has conjectured that $\theta = \frac{1}{2}$. It is easy to prove that $\theta \geqslant \frac{1}{2}$.

*Theorem* 1.4.2. In $\sigma > 1$,

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

where $\Lambda(1) = 0$. $\Lambda(p^m) = \log p$ and $\Lambda(n) = 0$ if $n \neq p^m$.

PROOF. Follows by Theorem 1.2.1.

Put

$$\sum_{n\leqslant x}\Lambda(n) = \psi(x), \quad \text{and} \quad \psi(x) = x + R_0(x).$$

Let $\theta'$ = infimum of those numbers $\alpha$ for which $\lim\limits_{x\to\infty} \frac{R_0(x)}{x^{\alpha}} = 0.$

Then we record some theorems without proof.

*Theorem* 1.4.3.

$$\theta' = \theta.$$

Similar theorem is also true of $\sum\limits_{p\leqslant x} \log p = \nu(x)$, $\sum\limits_{p\leqslant x} 1 = \pi(x)$ But for $\pi(x)$ we have to write $\pi(x) = \int\limits_2^x du/\log u + R(x)$ and modify $\theta'$ accordingly.

*Theorem* 1.4.4 (Hadamard and Vallée Poussin). [Explanation of $O$ (..): $f = O(g)$ means $|f|\,|g|^{-1}$ is bounded above.]

$$R(x) = O(xe^{-c\sqrt{\log x}}) \text{ with some } c > 0.$$

*Theorem* 1.4.5 (Littlewood)

$$R(x) = O(xe^{-c\sqrt{\log x \log\log x}}) \text{ with some } c > 0.$$

*Theorem* 1.4.6 (Vinogradov). For every fixed $\epsilon > 0$

$$R(x) = O_\epsilon\left(xe^{-\log x)^{\frac{3}{5}-\epsilon}}\right).$$

*Remark* : The result of Vinogradov is slightly more precise and reads

$$R(x) = O(x \exp(-c(\log x)^{3/5} (\log\log x)^{-1/5})).$$

*Theorem* 1.4.7. Let $N(\sigma, T)$ be the number of zeros of $\zeta(s)$ with real part between $\sigma$ and 1 (both inclusive) and imaginary part between 0 and $T$ (both inclusive). Suppose that for all $\sigma$ in $0 \leqslant \sigma \leqslant 1$ and all $T \geqslant 30$

$$N(\sigma, T) < T^{A(1-\sigma)} (\log T)^{300000}$$

(the exponent of $(\log T)$ is unimportant) where $A$ is a numerical constant. Denote by $p_n$ the $n$-th prime. Then

$$p_{n+1} - p_n = O_\epsilon(p_n^{1-1/A+\epsilon}).$$

*Remark.* This theorem is due to Ingham who also proved that we can take $A = 8/3$. Ingham also showed that if $\zeta(\frac{1}{2} + it) = O_\epsilon(t^\epsilon)$, $t \geqslant 10$ (a consequence of Riemann hypothesis) then $A = 2 + \epsilon$ for arbitrary $\epsilon$ would follow. Estimates for $N(\sigma, T)$ are called density estimates.

*Theorem* 1.4.8 (Ingham)

$$N(\sigma, T) = O(T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^{300000}).$$

*Theorem* 1.4.9 (Montgomery H. L. and Huxley)

$$N(\sigma, T) = O(T^{\frac{12(1-\sigma)}{5}} (\log T)^{300000}).$$

The functions $L(s, \chi) = \sum_{1}^{\infty} \chi(n) n^{-s}$ ($\chi$, character mod $q$) are also important [do not worry if you do not know character etc. you have still a bright future if you know how to tackle only $\zeta(s)$]. We can define $N_{\chi}(\sigma, T)$. We have an important theorem due to H. L. Montgomery, A. Selberg, Y. Motohashi and M. Jutila, *viz.*,

*Theorem* 1.4.10.

$$\sum_{q \leqslant Q} \sum_{\chi \bmod q}{}^{*} N_{\chi}(\sigma, T) = O_{\epsilon}((Q^2 T)^{(12/5+\epsilon)(1-\sigma)}).$$

(* denoting the restriction to primitive characters mod $q$).

*Remark*. The earlier estimate is due to H. L. Montgomery and M. N. Huxley (a simple proof was later found by M. Jutila), which is

$$O((Q^2 T)^{\frac{12}{5}(1-\sigma)} (\log (QT)^{3000000})).$$

As remarked earlier Riemann's conjecture implies $\zeta(\frac{1}{2} + it) = O_{\epsilon}(t^{\epsilon})$.

We record here some unconditional theorems without proof.

*Theorem* 1.4.11 (Hardy-Littlewood)

$$\zeta(\tfrac{1}{2} + it) = O(t^{1/6}).$$

*Theorem* 1.4.12 (R. Balasubramanian and D. H. Brown independently)

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, dt = T \log T + \beta T + O(T^{\frac{1}{3}})$$

where $\beta$ is a constant independent of $T$.

*Theorem* 1.4.13 (D. H.-Brown)

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 \, dt = T \sum_{\nu=0}^{4} \beta_{\nu} (\log T)^{4-\nu} + O_{\epsilon}(T^{7/8+\epsilon}).$$

$\beta_0, \ldots, \beta_4$ are independent of $T$.

## 2. Zero Free Regions

### 2.1. *A first result* $\zeta(1+it) \neq 0$.

*Theorem* 2.1.1. $\zeta(1+it) \neq 0$ for all real $t$.

FIRST PROOF. Since $\zeta(1) = \infty$ we may suppose that $t \neq 0$. Let $a$ be any real number and consider the inequality $(a + e^{i\theta} + e^{-i\theta})^2 \geqslant 0$. Note that for $\sigma > 1$, $\log \zeta(s) = \sum_{p} \sum_{m \geqslant 1} 1/mp^{ms}$, an absolutely convergent series.

We have

$$0 \leqslant \sum_{p} \sum_{m \geqslant 1} \frac{(a + p^{mit} + p^{-mit})^2}{mp^{m\sigma}}$$

$$= \sum \sum \frac{a^2 + 2 + 2a(p^{mit} + p^{-mit}) + p^{2mit} + p^{-2mit}}{mp^{m\sigma}}$$

$$= \log \{(\zeta(\sigma))^{a^2+2} \, | \zeta(\sigma + it)|^{4a} \, | \zeta(\sigma + 2it)|^2\}.$$

Thus

$$(\zeta(\sigma))^{a^2+2} \, | \zeta(\sigma + it)|^{4a} \, | \zeta(\sigma + 2it)|^2 \geqslant 1.$$

Suppose now that $\zeta(1+it) = 0$. The study of this inequality as $\sigma \to 1+0$ reveals that $4a \geqslant a^2 + 2$, which is false for instance if $a = 1$. This proves that $\zeta(1+it) \neq 0$.

SECOND PROOF. The starting point is now

$$-(a^2+2)\frac{\zeta'(\sigma)}{\zeta(\sigma)} - 2a\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} + \frac{\zeta'(\sigma-it)}{\zeta(\sigma-it)}\right)$$
$$-\left(\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} + \frac{\zeta'(\sigma-2it)}{\zeta(\sigma-2it)}\right) \geqslant 0.$$

Now

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} \sim \frac{1}{\sigma - 1}, \quad \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} \sim \frac{m}{\sigma - 1}$$

where $m$ is the order of zero at $\sigma = 1 + it$,

$$\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} \sim \frac{m'}{\sigma - 1}$$

where $m'$ is the order of zero at $\sigma = 1 + 2it$ and similar things at $1 - it$ and $1 - 2it$. We get $a^2 + 2 - 4a \geqslant 0$ ($a = 1$ is a contradiction as before).

*Remark* 1. The second proof is more powerful. The main step can be written

$$-(a^2+2)\frac{\zeta'(\sigma)}{\zeta(\sigma)}-4a\,\mathrm{Re}\,\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}-2\,\mathrm{Re}\,\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\geqslant 0.$$

We used rough estimate for the terms on the left. We can develop the argument further as will be seen.

*Remark* 2. We can instead of $(a+e^{i\theta}+e^{-i\theta})^2\geqslant 0$ use $10^8\,|\,1+z\,|^2+|\,1+z^2\,|^2\geqslant 3$ valid for all complex numbers $z$.

*Remark* 3. Hadamard, Vallée Poussin used $3+4\cos\theta+\cos 2\theta\geqslant 0$.

## 2.2. *Some estimate for* $\zeta(s)$

*Theorem* 2.2.1.

$$\zeta(s)=(O\,(1+t^{1-\sigma})\log t)\text{ for } t\geqslant 20 \text{ and } \sigma\geqslant\frac{1}{10}\quad\text{uniformly.}$$

PROOF. It is clear that if

$$X=[t],\ \sum_{n\leqslant X}n^{-s}=O\Big(\sum_{n\leqslant X}\frac{n^{1-\sigma}+1}{n}\Big)=O\,((1+t^{1-\sigma})\log t).$$

Now for $\sigma>0$,

$$\zeta(s)-\sum_{n\leqslant X}\frac{1}{n^s}=\sum_{n>X}\Big(\frac{1}{n^s}-\int_n^{n+1}\frac{du}{u^s}\Big)+\int_{X+1}^{\infty}\frac{du}{u^s}$$

$$=O\Big(\frac{(X+1)^{1-\sigma}}{t}+|\,s\,|\sum_{n>X}\int_n^{n+1}du\int_0^u\frac{dv}{v^{\sigma+1}}\Big)$$

$$=O\Big(t^{1-\sigma}+|\,s\,|\int_X^{\infty}\frac{du}{u^{\sigma+1}}\Big)=O\,(t^{1-\sigma})$$

since we can confine readily to $\sigma\leqslant 2$.

## 2.3. *Fundamental assumption on* $\zeta(s)$

We assume

$$\zeta(s)=O\,((t^{(1-\sigma)^a A})(\log t)^B)\ .\ 1\geqslant\sigma\geqslant\frac{1}{10};\ t\geqslant 20$$

($a$, $A$, $B$ are constants and these and other $O$-constants are independent of $\sigma$, $t$).

*Remark*. We have proved this assumption with $a = 1$, $A = 1$, $B = 1$. But I. M. Vinogradov has proved this with constants $a > 1$ and some other $A$, $B$. Best known $a$ is $a = 3/2$ due to Vinogradov's methods and is almost completely due to the ideas of Vinogradov.

## 2.4. *Landau's method*

We want a more precise bound for the terms

$$-\operatorname{Re}\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} \quad \text{and} \quad -\operatorname{Re}\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}.$$

Let $\rho_0 = \beta_0 + i\gamma_0$ be a zero of $\zeta(s)$ with $\beta_0 > 99/100$, $\gamma_0 > 2000$. (These constants are unimportant. 99/100 can be replaced by something between $\frac{1}{2}$ and 1). Put

$$M(\sigma,\tau) = \max_{a \geqslant \sigma,\ \tau/20 \leqslant t \leqslant 20\tau} |\zeta(a+it)|,$$

$$s_0 = \sigma_0 + i\gamma_0 \text{ with } 1 < \sigma_0 < \frac{1}{100} + 1.$$

Consider the disc $D : |s - s_0| \leqslant R < 1/100$. Either there is no zero of $\zeta(s)$ in this disc in which case $\beta_0 < \sigma_0 - R$ or there are zeros $\rho$ of $\zeta(s)$. In either case put

$$f(s) = \frac{\zeta(s)}{\prod_{\rho \in D}\left(1 - \frac{s - s_0}{\rho - s_0}\right)}.$$

Put $D_1 : |s - s_0| \leqslant 3R$. Then $\max_{s \in D_1} |f(s)| \leqslant \max_{s \in \mathrm{Bd} D_1} |f(s)| \leqslant M(1 - 3R, \gamma_0)$

since

$$\prod_{\rho \in D}\left|1 - \frac{s - s_0}{\rho - s_0}\right| \geqslant 1$$

on boundary Bd $D_1$ of $D_1$. Hence $f(s)$ is analytic in $D$ and also in $D_1$ and in $D$ we have

$$\operatorname{Re}\log f(s) \leqslant \log M(1 - 3R, \gamma_0) = \log M \text{ say.}$$

Also $\log f(s_0) = O(\log 1/(\sigma_0 - 1))$. Similarly we may prove results about the point $\beta_0 + 2i\gamma_0$ though of course $\beta_0 + 2i\gamma_0$ need not be a zero of $\zeta(s)$. We now need a lemma which we state as

*Theorem* 2.4.1. Let $f(z)$ be analytic in $|z - z_0| \leqslant R$ and on the boundary of this disc: $\operatorname{Re} F(z) \leqslant U$. Then

$$F'(z_0) = O\left(\frac{U + |F(z_0)|}{R}\right).$$

PROOF. Let $F(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$ with $a_j = |a_j| e^{i\alpha_j}$ whenever $a_j \neq 0, 0 \leqslant \alpha_j < 2\pi$. We have

$$\int_0^{2\pi} \mathrm{Re}\, F(z_0 + Re^{i\theta})\, d\theta = 2\pi\, \mathrm{Re}\, a_0,$$

and

$$\int_0^{2\pi} \mathrm{Re}\, F(z_0 + Re^{i\theta}) \cos(\theta + \alpha_1)\, d\theta = \pi\, |a_1|.$$

Thus

$$\pi |a_1| = \int_0^{2\pi} \mathrm{Re}\,(F(z_0 + Re^{i\theta}) - a_0) \cos(\theta + \alpha_1)\, d\theta$$

$$= \int_0^{2\pi} \mathrm{Re}\,(F(z_0 + Re^{i\theta}) - a_0)(1 + \cos(\theta + \alpha_1))\, d\theta$$

$$\leqslant 2(U - \mathrm{Re}\, a_0).$$

Therefore

$$|a_1| = |F'(z_0)| = O\left(\frac{U + |F(z_0)|}{R}\right).$$

As a corollary follows

$$\frac{f'(s_0)}{f(s_0)} = O\left(\frac{\log M + \log \frac{1}{\sigma_0 - 1}}{R}\right)$$

i.e.,

*Theorem* 2.4.2.

$$-\frac{\zeta'(s_0)}{\zeta(s_0)} + \sum_{\rho \in D} \frac{1}{s_0 - \rho} = O\left(\frac{\log M + \log \frac{1}{\sigma_0 - 1}}{R}\right),$$

in particular the same is true of the real part and so

$$-\mathrm{Re}\, \frac{\zeta'(s_0)}{\zeta(s_0)} + \sum_{\rho \in D} \mathrm{Re}\, \frac{1}{s_0 - \rho} = O\left(\frac{\log M + \log \frac{1}{\sigma_0 - 1}}{R}\right).$$

*Remark*. We observe that $\mathrm{Re}\, \dfrac{1}{s_0 - \rho} \geqslant 0$ and we record the corollary.

*Theorem* 2.4.3. Let $1 < \sigma_0 < 1 + \dfrac{1}{100}$ and $0 < R < \dfrac{1}{100}$. Then

$$-\mathrm{Re}\, \frac{\zeta'(\sigma_0 + 2i\gamma_0)}{\zeta(\sigma_0 + 2i\gamma_0)} < C\left(\frac{\log M + \log \frac{1}{\sigma_0 - 1}}{R}\right)$$

and

$$\begin{cases} \text{either } \beta_0 < \sigma_0 - R \\ \text{or } -\operatorname{Re} \dfrac{\zeta'(\sigma_0 + i\gamma_0)}{\zeta(\sigma_0 + i\gamma_0)} + \dfrac{1}{\sigma_0 - \beta_0} < C\left(\dfrac{\log M + \log \dfrac{1}{\sigma_0 - 1}}{R}\right) \end{cases}$$

where $C$ is a numerical constant.

We next put $\sigma_0 = 1 + \lambda(1 - \beta_0)$ where $\lambda$ is a large constant; accordingly $\lambda(1 - \beta_0) \leqslant 1/100$ is assumed (otherwise $1 - \beta_0 \geqslant 1/100\,\lambda$). We obtain, with $a = 1$ in the second proof of Theorem 2.2.1, either

$$\beta_0 < \sigma_0 - R, \text{ i.e., } \frac{1}{1-\beta_0} = O\left(\frac{1}{R}\right)$$

or

$$\begin{cases} \dfrac{1}{1-\beta_0} = O\left(\dfrac{\log M + \log \dfrac{1}{1-\beta_0}}{R}\right) \\ = O\left((R^{a-1} \log \gamma_0 + \dfrac{\log \dfrac{1}{1-\beta_0} + \log\log \gamma_0}{R}\right). \end{cases}$$

We may assume that $\log 1/(1-\beta_0) > \epsilon \log\log \gamma_0$ otherwise we have a zero free region $1 - \beta_0 \geqslant (\log \gamma_0)^{-\epsilon}$. So we have

$$\frac{1}{1-\beta_0} = O\left(R^{a-1} \log \gamma_0 + \frac{\log\left(\frac{1}{1-\beta_0}\right)}{R}\right).$$

If $a = 1$ we get the zero free region $1 - \beta_0 \geqslant \dfrac{C_1}{\log \gamma_0}$ by choosing $R = \dfrac{1}{100}$.

If $a > 1$ we minimise the R.H.S. subject to $0 < R \leqslant \dfrac{1}{100}$ and we get

$$\frac{1}{1-\beta_0} = O\left((\log \gamma_0)^{1/a} \left(\log\left(\frac{1}{1-\beta_0}\right)\right)^{1-1/a} + \log\left(\frac{1}{1-\beta_0}\right)\right).$$

This gives a zero free region. Summarising we have

*Theorem* 2.4.4. Let $\zeta(s) = O\left((t^{A(1-\sigma)^a})(\log t)^B\right)$ for $1 \geqslant \sigma \geqslant 1/10$ and $t \geqslant 100$. Then the zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with real part $\geqslant 1 - 1/100$ must satisfy $\beta < 1$ and further if $\gamma \geqslant 100$, we have,

$$\frac{1}{1-\beta} = O\left((\log \gamma)^{1/a} (\log\log \gamma)^{1-1/a}\right)$$

provided $a \geqslant 1$, $A$, $B$ are constants independent of $\sigma$ and $t$.

## 3. Density Results

### 3.1. *Outline of the method*

We put $N(\sigma, T_1, T_2) =$ number of zeros $\rho\,(= \beta + i\gamma)$ of $\zeta(s)$ subject to the conditions $\sigma \leqslant \beta < 1$, $T_1 \leqslant \gamma \leqslant T_2$. Next write $N(\sigma, T) = N(\sigma, 0, T)$. Our problem is to prove for $0 \leqslant \sigma < 1$ an estimate of the type

$$N(\sigma, T) < T^{\frac{3}{2-\sigma}(1-\sigma)} (\log T)^{50000}, \; T \geqslant 30.$$

We will prove only a weaker result where $3/(2-\sigma)$ is replaced by $4/(3-2\sigma)$. It suffices to prove a similar estimate for $N(\sigma, T, 2T)$. A preliminary step is

*Theorem* 3.1.1.

$$N(0, T, T+1) = O(\log T).$$

PROOF. The proofs of Theorems 1.3.1 and 1.3.2 show that

$$\max_{\sigma \geqslant -10, T/2 \leqslant t \leqslant 2T} |\zeta(s)| = O(T^{500}).$$

We apply maximum modulus principle to

$$F_1(s) = \frac{\zeta(s)}{\prod \left(1 - \dfrac{s-(2+iT)}{\rho-(2+iT)}\right)}$$

where $\rho$ runs over all the zeros of $\zeta(s)$ counted by $N(0, T, T+1)$. We have

$$|F_1(2+iT)| = O\left(\frac{T^{500}}{2^{N(0,T,T+1)}}\right).$$

Hence observing that $|F_1(2+iT)| \geqslant 1 - (\zeta(2)-1) = 2 - \zeta(2) > 0$ we have

$$N(0, T, T+1) = O(\log T)$$

$$\left(\text{observe that on } |s-2-iT| = 12, \left|\frac{s-2-iT}{\rho-2-iT}\right| \geqslant 3\right).$$

Next we divide the rectangle bounded by the lines with real parts $\sigma$, 1 and imaginary parts $T$, $2T$ into rectangles of height 1. We take the rectangles which contain a zero and count this number and multiply by $O(\log T)$ to get a bound for $N(\sigma, T, 2T)$. By applying a convexity argument we convert this problem into one of mean value upper bound for

$$\int_T^{2T} \left|(\zeta(\sigma+it))\left(\sum_{n \leqslant T} \frac{\mu(n)}{n^{\sigma+it}}\right) - 1\right|^{\tau} dt, (\sigma = \tfrac{1}{2}, 1).$$

$$(\tau = 1 \text{ if } \sigma = \tfrac{1}{2}, \tau = 2 \text{ if } \sigma = 1).$$

This leads to the solution of the problem.

3.2. *Mean value upper bounds*

Put $F_2(s) = \zeta(s) M_T(s) - 1$ where

$$M_T(s) = \sum_{n \leqslant T} \frac{\mu(n)}{n^s}.$$

Then we prove

*Theorem* 3.2.1.

$$\int_{T/2}^{3T} | F_2(\tfrac{1}{2} + it) | \, dt = O(T (\log T)^{10})$$

and

*Theorem* 3.2.2.

$$\int_{T/2}^{3T} | F_2 (1 + (\log T)^{-1} + it) |^2 \, dt = O((\log T)^{20}).$$

The key lemma (which is very simple) is

*Theorem* 3.2.3.

$$\int_{T/2}^{3T} | \sum_{n \leqslant X} a_n n^{it} |^2 \, dt = O((T + X \log X) \sum_{n \leqslant X} | a_n |^2).$$

To prove Theorem 3.2.3 we have only to observe that

$$\frac{1}{\log m/n} = O\left(\frac{m+n}{m-n}\right)$$

whenever $m \neq n$ and that

$$\sum_{m \neq n} \left| \frac{a_m \bar{a}_n}{m-n} \right| = O\left(\sqrt{\sum_{m \neq n} \frac{| a_m |^2}{| m-n |}} \sqrt{\sum_{m \neq n} \frac{| a_n |^2}{| m-n |}}\right).$$

In view of Theorem 3.1.1 it suffices to restrict in the bound for $N(\sigma, T, 2T)$ to $\sigma \geqslant \frac{1}{2} + 1/\log T$ since otherwise the estimate is trivial. Put $G(s) = F_2(s)$. Select a set of zeros in each of the rectangles which contain a zero $\rho$. Then by Cauchy's theorem we have

$$\int \frac{G(s) Y^{s-\rho} e^{(s-\rho)^2}}{s-\rho} \, ds = 2\pi i \text{ times the multiplicity of } \rho$$

the integral being taken over the rectangle $\rho + x + iy$, $\frac{1}{2} < \beta + x < 1$, $| \gamma + y | < (\log T)^2$. If $\log Y = O(\log T)$ then the contributions from the horizontal sides is $O(T^{-10})$. Denoting the vertical sides by $V_1$, $V_2$ we have

$$1 = O(\log T (\int_{V_1} | G(s) | \, dt) Y^{\frac{1}{2}-\beta} + \log T (\int_{V_2} | G(s) | \, dt) Y^{1-\beta}).$$

We can replace $\int\limits_{V_1}$ by $1+\int\limits_{V_1}$ and $\int\limits_{V_2}$ by $T^{-10}+\int\limits_{V_2}$ and fix $Y$ by $((1+\int\limits_{V_1})(T^{-10}+\int\limits_{V_2})^{-1})^2$.

This satisfies $\log Y = O(\log T)$ and we get

*Theorem* 3.2.4.

$$1 = O\Big((\log T)\big((1+\int\limits_{V_1})^{2(1-\beta)}(T^{-10}+\int\limits_{V_2})^{2\beta-1}\big)\Big).$$

By Theorem 3.1.1 and 3.2.2,

*Theorem* 3.2.5.

$$\sum_{\rho}(1+\int\limits_{V_1}) = O(T(\log T)^{15})$$

and

$$\sum_{\rho}(T^{-10}+\int\limits_{V_2})^2 = O((\log T)^{30}).$$

Next the number of zeros with

$$1+\int\limits_{V_1} \geqslant W_1 \text{ is } O\left(\frac{T(\log T)^{15}}{W_1}\right)$$

and of those with

$$T^{-10}+\int\limits_{V_2} \geqslant W_2 \text{ is } = O\left(\frac{(\log T)^{30}}{W_2^2}\right).$$

For the remaining zeros we have by Theorem 3.2.4, $1=O((\log T)\, W_1^{2(1-\beta)}\, W_2^{2\beta-1})$. We next fix $W_1^{2(1-\sigma)}\, W_2^{2\sigma-1} = (\log T)^{-1} C$ where $C$ is a small constant. Then $N(\sigma, T, 2T) = O(\log T)^{40}(T/W_1 + 1/W_2^2)$. We next put $W_1 = W_2^2\, T$ and get $T^{2(1-\sigma)}$ $W_2^{3-2\sigma} = (\log T)^{-1} C$ so that $N(\sigma, T, 2T) = O((\log T)^{40}(\log T)^{-1} C/T^{2(1-\sigma)})^{-2/(3-2\sigma)}$. This gives what we want since $\beta \geqslant \sigma$ and $W_2 \geqslant T^{-\frac{1}{2}}$ can be assumed.

3.3. *Proof of Theorem* 3.2.1. We apply Hölder's inequality and we see that we have only to prove

*Theorem* 3.3.1. $\int\limits_{T/2}^{3T} |\zeta(\frac{1}{2}+it)|^4\, dt = O(T(\log T)^4)$.

In view of

$$\zeta(s)(1-2^{1-s}) = \sum_{1}^{\infty} n^{-s}(-1)^{n-1}$$

we have only to prove

*Theorem* 3.3.2.

$$\int\limits_{T/2}^{3T} \Big|\sum_{X \leqslant n \leqslant 2X} (-1)^{n-1} n^{-\frac{1}{2}-it}\Big|^4 dt = O(T(\log T)^3) \quad \text{for } X \geqslant T.$$

PROOF.

$$\text{L.H.S.} = O\left(\int_{T/2}^{3T} dt \left( \left| \sum_{X/2 \leqslant n \leqslant X} \left(\tfrac{1}{2} + it\right) \int_{2n}^{2n+1} \frac{du}{u^{3/2+it}} \right) \right|^2 \right)$$

$$= O\left(T^2 \int_{T/2}^{3T} dt \left| \int_0^{1/2} du \sum_{X/2 \leqslant n \leqslant X} \frac{1}{(n+u)^{3/2+it}} \right|^2 \right)$$

$$= O\left(T^2 \int_0^{1/2} du \int_{T/2}^{3T} \left| \sum_{X/2 \leqslant n \leqslant X} \frac{1}{(n+u)^{3/2+it}} \right|^2 dt \right)$$

$$= O\,(T \log T)$$

by arguments similar to the proof of Theorem 3.2.3.

3.4. *Proof of Theorem* 3.2.2. We have only to observe that

$$\zeta\left(1 + \frac{1}{\log T} + it\right) = \sum_{n \leqslant T^{50}} \frac{1}{n^{1+(1/\log T)+it}} + O\,(T^{-1})$$

(by using the argument of the proof of Theorem 1.3.2). Next we have only to prove that

$$\int_{T/2}^{3T} \left| \left( \sum_{n \leqslant T^{50}} n^{-1-(1/\log T)-it} \right) \left( \sum_{n \leqslant T} \mu(n)\, n^{-1-(1/\log T)-it} \right) - 1 \right|^2 dt$$

$$= O\,((\log T)^{10}).$$

This follows from the fact that $\sum_{n \leqslant x} (d(n))^2 = O\,(x\,(\log x)^4)$. This can be seen as follows $(d(n))^2 < d_4(n)$, where $(\zeta(s))^4 = \sum_1^{\infty} d_4(n)\, n^{-s}$. For

$$(a+1)^2 < \left| \binom{-4}{a} \right| = \left| \frac{-4 - 5 \ldots -(a+4-1)}{a!} \right|$$

$$= \frac{(a+1)(a+2)(a+3)}{3\,!}$$

(Since $6a + 6 < a^2 + 5a + 6$, i.e., $a < a^2$)

and

$$\sum_{n \leqslant x} (d(n))^2 < x^{1+(1/\log x)} \sum_1^{\infty} n^{-1-(1/\log x)}\, d_4(n) = O\,(x\,(\log x)^4).$$

## Suggestions for Further Reading


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## APPENDIX

*(added in proof)*

We would like to leave some exercises for an enthusiast. Starting with the series $f(t) = \sum_{n=1}^{\infty} ((-1)^{n-1} n^{-\frac{1}{2}-it}) = \zeta(\frac{1}{2} + it)(1 - 2^{\frac{1}{2}-it})^{-1}$, $t \geq 2$, it is trivial to prove $f(t) = O(t)$ and in fact $f(t) = O(t^{\frac{1}{2}})$. These things need the estimates for $S$ defined by $S = \sum_{a \leq n \leq a+h} (-1)^{n-1} n^{it}$ where $a \geq 1$, $1 \leq h \leq a$ (see the equation 5.2.1 of Titchmarsh's book). The trivial estimate $S = O(h)$ is enough to prove $f(t) = O(t^{\frac{1}{2}})$. Actually from the arguments of lemma 5.3 with $k = 1$ and from a routine imitation of the arguments on pages 85 and 86 it follows that $S = O(\sqrt{t})$. From this and equation 5.2 it follows that $\sum_{n \geq x} (-1)^{n-1} n^{-\frac{1}{2}-it} = O((t/x)^{\frac{1}{2}} (\log t)^2)$ for $t \geq 2, x \geq 1$. We now take $x = t^{\frac{1}{2}}$, then this tail portion is $O(t^{\frac{1}{4}} (\log t)^2)$. Also $\sum_{n \leq x} (-1)^{n-1} n^{-\frac{1}{2}-it} = O(t^{\frac{1}{4}})$ and so $f(t) = O(t^{\frac{1}{4}} (\log t)^2)$. But taking $x = t^{\frac{2}{3}}$ the tail portion is $O(t^{1/6} (\log t)^2)$. The arguments on pages 85 and 89 (with $k = 2$) show that $\sum_{n \leq t^{2/3}} (-1)^{n-1} n^{-\frac{1}{2}-it} = O(t^{1/6} (\log t)^2)$. Thus $f(t) = O(t^{1/6} (\log t)^2)$. All these follow straight from the definition of $f(t)$ and involve only simple calculus. [We do not need functional equation for $\zeta(s)$, estimates for $\Gamma(s)$ and so on.] Try to study Chapters V and VI of Titchmarsh's book and try to prove $f(t) = O(t^{1/7})$. It may be mentioned that the best known estimate is a very poor but a difficult improvement $f(t) = O(t^{\frac{173}{1067}} (\log t)^2)$ on $f(t) = O(t^{1/6})$. I wish you all good luck in the solution of this problem.

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